



k-Product Cordial Labeling of Splitting Graph of Star Graphs

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Abstract

Let f be a map from $V(G)$ to $\{0, 1, \dots, k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v) \pmod k$. f is called a k -product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, \dots, k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x ($x = 0, 1, \dots, k-1$). A graph that admits k -product cordial labeling is called k -product cordial graph. We have already proved that several families of graphs admit k -product cordial labeling. In this paper, we show that the splitting graph of star graphs admit k -product cordial labeling.

Keywords: cordial labeling, product cordial labeling, k -product cordial labeling, splitting graph, star graph.

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1. Introduction and Terminology

All graphs considered here are simple, finite, connected and undirected. We follow the basic notations and terminology of graph theory as in [3]. The concept of labeling of graph has gained a lot of popularity in the field of graph theory. In 1967, Rosa [14] published a pioneering paper on graph labeling problems. Since then, many graph labeling techniques have been introduced and studied by several authors (refer [2]). Cordial labeling is one of the popular labelings defined by Cahit [1] in 1987: Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Followed by this, in 2004 Sundaram et al. [15] introduced a variation of cordial labeling called product cordial labeling: Let f be a function from $V(G)$ to $\{0, 1\}$. For each edge uv , assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with i ($i = 0, 1$). Several papers have been published on this topic.

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Later in 2012, Ponraj et al. [13] further extended the concept of product cordial labeling and defined k -product cordial labeling as follows: Let f be a map from $V(G)$ to $\{0, 1, \dots, k-1\}$ where k is an integer, $1 \leq k \leq |V(G)|$. For each edge uv assign the label $f(u)f(v)(\text{mod } k)$. f is called a k -product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$, and $|e_f(i) - e_f(j)| \leq 1$, $i, j \in \{0, 1, \dots, k-1\}$, where $v_f(x)$ and $e_f(x)$ denote the number of vertices and edges respectively labeled with x ($x = 0, 1, \dots, k-1$). A graph that admits k -product cordial labeling is called k -product cordial graph. We have already proved that several families of graphs admit k -product cordial labeling. The concept of k -product cordial labeling and the work of the authors in [13], motivated us to do further research on this topic. Consequently in our research, we established that the following families of graphs admit k -product cordial labeling: union of graphs [4]; cone and double cone graphs [5]; Napier bridge graphs [6]; product of graphs [7]; powers of paths [8]; fan and double fan graphs [9]; the maximum number of edges in a 4-product cordial graph of order p is $4\lceil \frac{p-1}{4} \rceil \lfloor \frac{p-1}{4} \rfloor + 3$ [10] and path graphs [11]. Jeyanthi et al. [12] already proved that the splitting graph $S'(K_{1,n})$ is a 3-product cordial graph. In this paper, we show that splitting graph of star graphs admit k -product cordial labeling for $k \geq 4$.

We use the following terminology to prove our main results. The splitting graph $S'(G)$ of a graph G is a graph obtained by adding a new vertex v' to each vertex v of G such that v' is adjacent to every vertex that is adjacent to v in G . A bipartite graph is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2 . If every vertex of V_1 is adjacent with every vertex of V_2 , then G is a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. The graph $K_{1,n}$ is called a star graph. An illustration of $S'(K_{1,4})$ is shown in Figure 1.

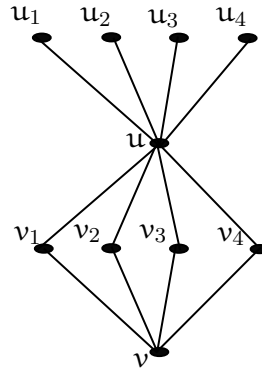


Figure 1: Splitting graph of star graph $K_{1,4}$

2. Main Results

Theorem 2.1. The graph $S'(K_{1,n})$ is k -product cordial for $n \geq \frac{k}{2}$ where k is even and $k \geq 4$.

Proof. Let the vertex set and the edge set of $S'(K_{1,n})$ be $V(S'(K_{1,n})) = \{u, v, u_i, v_i ; 1 \leq i \leq n\}$ and $E(S'(K_{1,n})) = \{(u, u_i) ; 1 \leq i \leq n\} \cup \{(v, v_i) ; 1 \leq i \leq n\} \cup \{(u, v_i) ; 1 \leq i \leq n\}$ respectively. We have the following four cases.

Define $f : V(S'(K_{1,n})) \rightarrow \{0, 1, 2, \dots, k-1\}$ for $k \geq 4$ as follows:

Case (i): If $n \equiv 0(\text{mod } k)$, then

$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i(\text{mod } k)$ and $f(v_i) = i(\text{mod } k)$ for $1 \leq i \leq n$.

Case (ii): If $n \equiv k-1(\text{mod } k)$, then

$f(u) = 1$, $f(u_{n-1}) = 0$, $f(u_n) = 2$, $f(v) = k-1$, $f(v_{n-1}) = 0$, $f(v_n) = 1$, $f(u_i) = i(\text{mod } k)$ and $f(v_i) = i(\text{mod } k)$ for $1 \leq i \leq \lfloor \frac{n}{k} \rfloor k$.

For $i = \lfloor \frac{n}{k} \rfloor k + j$; $1 \leq j \leq k-3$,

$f(u_i) = j+1$ and $f(v_i) = j+2$.

Case (iii): If $n \equiv x(\text{mod } k)$; $1 \leq x \leq \frac{k}{2} - 1$, then

$f(u) = 1, f(v) = k - 1, f(u_i) = i \pmod k$ and $f(v_i) = i \pmod k$ for $1 \leq i \leq k(\lfloor \frac{n}{k} \rfloor - 1)$.

For $i = k(\lfloor \frac{n}{k} \rfloor - 1) + j; 1 \leq j \leq 2x$,

$$f(u_i) = \begin{cases} \frac{k}{2} & \text{if } j = k - 3 \\ 0 & \text{if } j = k - 2. \end{cases}$$

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ k - 1 - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k(\lfloor \frac{n}{k} \rfloor - 1) + 2x + j; 1 \leq j \leq n - (k(\lfloor \frac{n}{k} \rfloor - 1) + 2x)$,

$$f(u_i) = \begin{cases} 0 & \text{if } j = k - 2 \\ 1 & \text{if } j = k - 1. \end{cases}$$

$$f(u_i) = \begin{cases} k - 1 - \frac{j+1}{2} & \text{if } j \text{ is odd} \\ 1 + \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k(\lfloor \frac{n}{k} \rfloor - 1) + j; 1 \leq j \leq k - 3, f(v_i) = j + 1$.

For $i = k\lfloor \frac{n}{k} \rfloor - 3 + j; 1 \leq j \leq n - (k\lfloor \frac{n}{k} \rfloor - 3)$,

$$f(v_i) = \begin{cases} k - 1 & \text{if } j = 1, 4 \\ 1 & \text{if } j = 2, 5 \\ 0 & \text{if } j = 3, 6 \\ \frac{k}{2} & \text{if } j = 7. \end{cases}$$

$$\text{If } j \geq 8, \text{ then } f(v_i) = \begin{cases} \frac{k}{2} - \frac{j-6}{2} & \text{if } j \text{ is even} \\ \frac{k}{2} + \frac{j-7}{2} & \text{if } j \text{ is odd.} \end{cases}$$

Case (iv): If $n \equiv k - x \pmod k; 2 \leq x \leq \frac{k}{2}$, then

$f(u) = 1, f(v) = k - 1, f(u_i) = i \pmod k$ and $f(v_i) = i \pmod k$ for $1 \leq i \leq k\lfloor \frac{n}{k} \rfloor$.

For $i = k\lfloor \frac{n}{k} \rfloor + j; 1 \leq j \leq k - 2x$,

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ k - 1 - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + k - 2x + j; 1 \leq j \leq x$,

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ k - 1 - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + j; 1 \leq j \leq 4$,

$$f(v_i) = \begin{cases} 0 & \text{if } j = 1 \\ 1 & \text{if } j = 2 \\ k - 1 & \text{if } j = 3 \\ \frac{k}{2} & \text{if } j = 4. \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + 4 + j; 1 \leq j \leq n - (k\lfloor \frac{n}{k} \rfloor + 4)$,

$$f(v_i) = \begin{cases} \frac{k}{2} + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ \frac{k}{2} - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

From the above cases, we have

for $n \equiv x \pmod k; x = \frac{k}{2} - 1$ or $k - 1, v_f(i) = \frac{2n+2}{k}$ for $0 \leq i \leq k - 1$.

If $n \equiv x \pmod k; 0 \leq x \leq \frac{k}{2} - 2$, then

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } 1 \leq i \leq x + 1, k - (x + 1) \leq i \leq k - 1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \text{ or } x + 2 \leq i \leq k - (x + 2). \end{cases}$$

If $n \equiv (\frac{k}{2} + x) \pmod k; 0 \leq x \leq \frac{k}{2} - 2$ and $k > 4$, then

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } 1 \leq i \leq x + 1, k - (x + 1) \leq i \leq k - 1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \text{ or } x + 2 \leq i \leq k - (x + 2). \end{cases}$$

If $k = 4$ and $n \equiv 2 \pmod 4$, then

$$v_f(i) = \begin{cases} 2\lfloor \frac{n}{k} \rfloor + 2 & \text{if } i = 1, 2 \\ 2\lfloor \frac{n}{k} \rfloor + 1 & \text{otherwise.} \end{cases}$$

Case 1: If $n \equiv 0 \pmod{k}$, then $e_f(i) = \frac{3n}{k}$ for $0 \leq i \leq k-1$.

Case 2: If $n \equiv k-1 \pmod{k}$, then

$$e_f(i) = \begin{cases} 3\lfloor \frac{n}{k} \rfloor + 2 & \text{if } i = 1, k-1, k-2 \\ 3\lfloor \frac{n}{k} \rfloor + 3 & \text{otherwise.} \end{cases}$$

Case 3: If $n \equiv x \pmod{k}$; $1 \leq x \leq \lfloor \frac{k}{3} \rfloor$; $k > 4$, then for $x = 1$

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 1, 2, k-2 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $k = 4$ and $x = 1$, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, 2 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $k = 6$ and $x = 2$,

$$e_f(i) = 3(\lfloor \frac{n}{k} \rfloor - 1) + 4; 0 \leq i \leq 5,$$

for $k > 6$ and $x = 2$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 1, 2, 3, k-1, k-2, k-3 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $x = 3$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, 2, 3, 4, k-1, k-2, k-3, k-4 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, \dots, x+1, k-(x+1), \\ & \frac{k}{2}, \frac{k}{2}+1, \frac{k}{2}-1, \dots, \frac{k}{2}-(\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2}+(\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, \dots, x+1, k-(x+1), \\ & \frac{k}{2}, \frac{k}{2}+1, \frac{k}{2}-1, \dots, \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

Subcase 1: If $n \equiv x \pmod{k}$ where $k \equiv 0 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \frac{k}{2} - 2$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), \dots, x+1, k-(x+1), \\ & \frac{k}{2}+(\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2}-(\lfloor \frac{k}{6} \rfloor - 1), \dots, \frac{k}{2}-(\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2}+(\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \frac{k}{2} - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), \dots, x+1, k-(x+1), \\ & \frac{k}{2}+(\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2}-(\lfloor \frac{k}{6} \rfloor - 1), \dots, \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv x \pmod{k}$ where $k \equiv 0 \pmod{3}$, $x = \frac{k}{2} - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), \dots, x, k-x, x+1, 0, \\ & \frac{k}{2}+(\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2}-(\lfloor \frac{k}{6} \rfloor - 1), \dots, \frac{k}{2}-(\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2}+(\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \frac{k}{2} - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3}+2, k-(\frac{k}{3}+2), \frac{k}{3}+3, k-(\frac{k}{3}+3), \dots, x, k-x, x+1, 0, \\ & \frac{k}{2}+(\lfloor \frac{k}{6} \rfloor - 1), \frac{k}{2}-(\lfloor \frac{k}{6} \rfloor - 1), \dots, \frac{k}{2}+(\frac{x}{2}-2), \frac{k}{2}-(\frac{x}{2}-2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 3: If $n \equiv x \pmod{k}$ where $k \equiv 1 \pmod{3}$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = x+1, k-(x+1), \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv x \pmod{k}$ where $k \equiv 1 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2} - 2$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k-(\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k-(\lfloor \frac{k}{3} \rfloor + 3), \dots, x+1, k-(x+1), \\ & \frac{k}{2}+(\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \frac{k}{2}-(\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \dots, \frac{k}{2}-(\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2}+(\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2} - 2$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \dots, x+1, k - (x+1), \\ & \frac{k}{2} + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \frac{k}{2} - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \dots, \frac{k}{2} + (\frac{x}{2} - 2), \frac{k}{2} - (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv x \pmod{k}$ where $k \equiv 1 \pmod{3}$, $x = \frac{k}{2} - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \dots, x, k - x, x+1, 0, \\ & \frac{k}{2} + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \frac{k}{2} - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \dots, \frac{k}{2} - (\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2} + (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \frac{k}{2} - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \dots, x, k - x, x+1, 0, \\ & \frac{k}{2} + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \frac{k}{2} - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \dots, \frac{k}{2} + (\frac{x}{2} - 2), \frac{k}{2} - (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 6: If $n \equiv x \pmod{k}$ where $k \equiv 2 \pmod{3}$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k - (\lfloor \frac{k}{3} \rfloor + 2), \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 7: If $n \equiv x \pmod{k}$ where $k \equiv 2 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2} - 2$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \dots, x+1, k - (x+1), \\ & \frac{k}{2} - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \frac{k}{2} + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \dots, \frac{k}{2} - (\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2} + (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2} - 2$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \dots, x+1, k - (x+1), \\ & \frac{k}{2} - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \frac{k}{2} + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \dots, \frac{k}{2} + (\frac{x}{2} - 2), \frac{k}{2} - (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 8: If $n \equiv x \pmod{k}$ where $k \equiv 2 \pmod{3}$, $x = \frac{k}{2} - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \dots, x, k - x, x+1, 0, \\ & \frac{k}{2} - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \frac{k}{2} + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \dots, \frac{k}{2} - (\lfloor \frac{x}{2} \rfloor - 2), \frac{k}{2} + (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \frac{k}{2} - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = k - (\lfloor \frac{k}{3} \rfloor + 2), \lfloor \frac{k}{3} \rfloor + 3, k - (\lfloor \frac{k}{3} \rfloor + 3), \dots, x, k - x, x+1, 0, \\ & \frac{k}{2} - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor - 1), \frac{k}{2} + (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor), \dots, \frac{k}{2} + (\frac{x}{2} - 2), \frac{k}{2} - (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 9: If $k = 8$ and $n \equiv 3 \pmod{8}$, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = 0 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Case 4: If $n \equiv k - x \pmod{k}$; $2 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is odd, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{if } i = \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} - 1, \dots, \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & 0, k-1, 1, k-2, 2, \dots, (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 3 & \text{otherwise.} \end{cases}$$

for $2 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{if } i = \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} - 1, \dots, \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & 0, k-1, 1, k-2, 2, \dots, k - (\frac{x}{2} + 1), \frac{x}{2} + 1 \\ 3(\lfloor \frac{n}{k} \rfloor) + 3 & \text{otherwise.} \end{cases}$$

Subcase 1: If $n \equiv k - x \pmod{k}$ where $k \equiv 0 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \frac{k}{2}$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\frac{k}{3} - 1), \frac{k}{2} - (\frac{k}{3} - 1), \frac{k}{2} + \frac{k}{3}, \frac{k}{2} - \frac{k}{3}, \dots, \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{k}{6} \rfloor + 2), (\lfloor \frac{k}{6} \rfloor + 2), \dots, (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \frac{k}{2}$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\frac{k}{3} - 1), \frac{k}{2} - (\frac{k}{3} - 1), \frac{k}{2} + \frac{k}{3}, \frac{k}{2} - \frac{k}{3}, \dots, \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{k}{6} \rfloor + 2), (\lfloor \frac{k}{6} \rfloor + 2), \dots, k - (\frac{x}{2} + 1), (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv k - x \pmod{k}$ where $k \equiv 1 \pmod{3}$ and $k > 4$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 3: If $k = 4$, $n \equiv 2 \pmod{4}$, then

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = 1, 3 \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv k - x \pmod{k}$ where $k \equiv 1 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2}$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \frac{k}{2} - \lfloor \frac{k}{3} \rfloor, \\ & \dots, \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), \dots, (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2}$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \frac{k}{2} - \lfloor \frac{k}{3} \rfloor, \\ & \dots, \frac{k}{2} + (x-2), \frac{k}{2} - (x-2), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), \dots, k - (\frac{x}{2} + 1), (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv k - x \pmod{k}$ where $k \equiv 2 \pmod{3}$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 6: If $n \equiv k - x \pmod{k}$ where $k \equiv 2 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2}$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \dots, \frac{k}{2} - (x-2), \frac{k}{2} + (x-2), \\ & (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), \\ & \dots, (\lfloor \frac{x}{2} \rfloor + 1), k - (\lfloor \frac{x}{2} \rfloor + 2) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \frac{k}{2}$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \frac{k}{2} + (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} - (\lfloor \frac{k}{3} \rfloor - 1), \frac{k}{2} + \lfloor \frac{k}{3} \rfloor, \dots, \frac{k}{2} - (x-2), \frac{k}{2} + (x-2), \\ & (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 3), \dots, k - (\frac{x}{2} + 1), (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Therefore, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq k-1$. Hence, the graph $S'(K_{1,n})$ is k -product cordial for $n \geq \frac{k}{2}$ where k is even and $k \geq 4$. $\square$

An example of 8-product cordial labeling of $S'(K_{1,10})$ shown in Figure 2.

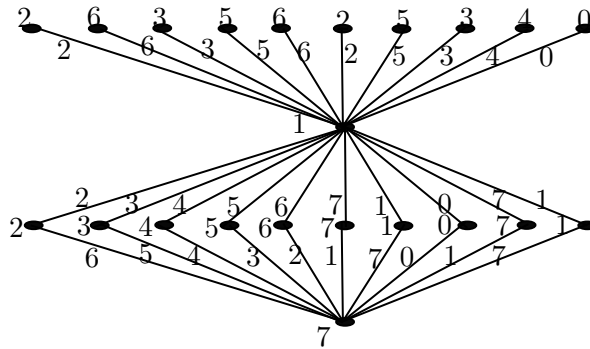


Figure 2: 8-Product cordial labeling of $S'(K_{1,10})$

Theorem 2.2. The graph $S'(K_{1,n})$ is k -product cordial for $n \geq \lfloor \frac{k}{2} \rfloor$ where k is odd and $k \geq 4$.

Proof. Let the vertex set and the edge set of $S'(K_{1,n})$ be $V(S'(K_{1,n})) = \{u, v, u_i, v_i ; 1 \leq i \leq n\}$ and $E(S'(K_{1,n})) = \{(u, u_i) ; 1 \leq i \leq n\} \cup \{(v, v_i) ; 1 \leq i \leq n\} \cup \{(u, v_i) ; 1 \leq i \leq n\}$ respectively. We have the following four cases.

Define $f : V(S'(K_{1,n})) \rightarrow \{0, 1, 2, \dots, k-1\}$ for $k \geq 4$ as follows:

Case (i): If $n \equiv 0 \pmod{k}$, then

$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i \pmod{k}$ and $f(v_i) = i \pmod{k}$ for $1 \leq i \leq n$.

Case (ii): If $n \equiv k-1 \pmod{k}$, then

$f(u) = 1$, $f(u_{n-1}) = 0$, $f(u_n) = 0$, $f(v) = k-1$.

$f(u_i) = i \pmod k$ and $f(v_i) = i \pmod k$ for $1 \leq i \leq \lfloor \frac{n}{k} \rfloor k$.

For $i = \lfloor \frac{n}{k} \rfloor k + j$; $1 \leq j \leq k-3$, $f(u_i) = j+1$.

For $i = \lfloor \frac{n}{k} \rfloor k + j$; $1 \leq j \leq k-1$, $f(v_i) = j$.

Case (iii): If $n \equiv x \pmod k$; $1 \leq x \leq \lfloor \frac{k}{2} \rfloor$, then

$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i \pmod k$ and $f(v_i) = i \pmod k$ for $1 \leq i \leq k(\lfloor \frac{n}{k} \rfloor - 1)$.

For $i = k(\lfloor \frac{n}{k} \rfloor - 1) + j$; $1 \leq j \leq 2x$,

$$f(u_i) = \begin{cases} 0 & \text{if } j = 2\lfloor \frac{k}{2} \rfloor - 1 \\ 1 & \text{if } j = 2\lfloor \frac{k}{2} \rfloor. \end{cases}$$

$$f(u_i) = \begin{cases} k-1 - \frac{j+1}{2} & \text{if } j \text{ is odd} \\ 1 + \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k(\lfloor \frac{n}{k} \rfloor - 1) + 2x + j$; $1 \leq j \leq n - (k(\lfloor \frac{n}{k} \rfloor - 1) + 2x)$,

$$f(u_i) = \begin{cases} 0 & \text{if } j = k-2 \\ 1 & \text{if } j = k-1. \end{cases}$$

$$f(u_i) = \begin{cases} k-1 - \frac{j+1}{2} & \text{if } j \text{ is odd} \\ 1 + \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k(\lfloor \frac{n}{k} \rfloor - 1) + j$; $1 \leq j \leq k-3$, $f(v_i) = 1+j$.

For $i = k\lfloor \frac{n}{k} \rfloor - 3 + j$; $1 \leq j \leq n - (k\lfloor \frac{n}{k} \rfloor - 3)$,

$$f(v_i) = \begin{cases} k-1 & \text{if } j = 1, 4 \\ 1 & \text{if } j = 2, 5 \\ 0 & \text{if } j = 3, 6. \end{cases}$$

$$\text{If } j \geq 7, \text{ then } f(v_i) = \begin{cases} \lfloor \frac{k}{2} \rfloor + 1 - \frac{j-5}{2} & \text{if } j \text{ is odd} \\ \lfloor \frac{k}{2} \rfloor + \frac{j-6}{2} & \text{if } j \text{ is even.} \end{cases}$$

Case (iv): If $n \equiv k-x \pmod k$; $2 \leq x \leq \lfloor \frac{k}{2} \rfloor$, then

$f(u) = 1$, $f(v) = k-1$, $f(u_i) = i \pmod k$ and $f(v_i) = i \pmod k$ for $1 \leq i \leq k\lfloor \frac{n}{k} \rfloor$.

For $i = k\lfloor \frac{n}{k} \rfloor + j$; $1 \leq j \leq 2$, $f(u_i) = 0$.

For $i = k\lfloor \frac{n}{k} \rfloor + 2 + j$; $1 \leq j \leq 2(\lfloor \frac{k}{2} \rfloor - x)$,

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ k-1 - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + 2 + 2(\lfloor \frac{k}{2} \rfloor - x) + j$; $1 \leq j \leq n - (k\lfloor \frac{n}{k} \rfloor + 2 + 2(\lfloor \frac{k}{2} \rfloor - x))$,

$$f(u_i) = \begin{cases} 1 + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ k-1 - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

$$\text{For } i = k\lfloor \frac{n}{k} \rfloor + j; 1 \leq j \leq 2, f(v_i) = \begin{cases} 1 & \text{if } j = 1 \\ k-1 & \text{if } j = 2. \end{cases}$$

For $i = k\lfloor \frac{n}{k} \rfloor + 2 + j$; $1 \leq j \leq n - (k\lfloor \frac{n}{k} \rfloor + 2)$,

$$f(v_i) = \begin{cases} \lfloor \frac{k}{2} \rfloor + \frac{j+1}{2} & \text{if } j \text{ is odd} \\ \lfloor \frac{k}{2} \rfloor + 1 - \frac{j}{2} & \text{if } j \text{ is even.} \end{cases}$$

From the above cases, we have

for $n \equiv \lfloor \frac{k}{2} \rfloor \pmod k$,

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } i = 1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \text{ or } 2 \leq i \leq k-1. \end{cases}$$

If $n \equiv x \pmod k$; $0 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$, then

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } 1 \leq i \leq x+1, k-(x+1) \leq i \leq k-1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } i = 0 \text{ or } x+2 \leq i \leq k-(x+2). \end{cases}$$

If $n \equiv k-1 \pmod k$, then $v_f(i) = \frac{2n+2}{k}$ for $0 \leq i \leq k-1$.

If $n \equiv (\lfloor \frac{k}{2} \rfloor + x) \pmod k$; $1 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$, then

$$v_f(i) = \begin{cases} \lfloor \frac{2n+2}{k} \rfloor + 1 & \text{if } 0 \leq i \leq x, k-x \leq i \leq k-1 \\ \lfloor \frac{2n+2}{k} \rfloor & \text{if } x+1 \leq i \leq k-(x+1). \end{cases}$$

Case 1: If $n \equiv 0 \pmod{k}$, then $e_f(i) = \frac{3n}{k}$ for $0 \leq i \leq k-1$.

Case 2: If $n \equiv k-1 \pmod{k}$, then

$$e_f(i) = \begin{cases} 3\lfloor \frac{n}{k} \rfloor + 2 & \text{if } i = 0, 1, k-1 \\ 3\lfloor \frac{n}{k} \rfloor + 3 & \text{otherwise.} \end{cases}$$

Case 3: If $n \equiv x \pmod{k}$; $1 \leq x \leq \lfloor \frac{k}{3} \rfloor$, then for $x = 1$

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 1, 2, k-2 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $x = 2$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 1, 2, 3, k-1, k-2, k-3 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $x = 3$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, 2, 3, 4, k-1, k-2, k-3, k-4 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, \dots, x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil - 1, \lceil \frac{k}{2} \rceil + 1, \dots, \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

for $4 \leq x \leq \lfloor \frac{k}{3} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{if } i = 0, 1, k-1, 2, k-2, \dots, x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil - 1, \lceil \frac{k}{2} \rceil + 1, \dots, \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 3 & \text{otherwise.} \end{cases}$$

Subcase 1: If $n \equiv x \pmod{k}$ where $k \equiv 0 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3} + 2, k - (\frac{k}{3} + 2), \frac{k}{3} + 3, k - (\frac{k}{3} + 3), \dots, x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{k}{6} \rfloor - 1), \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{6} \rfloor, \lceil \frac{k}{2} \rceil + \lfloor \frac{k}{6} \rfloor, \\ & \dots, \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3} + 2, k - (\frac{k}{3} + 2), \frac{k}{3} + 3, k - (\frac{k}{3} + 3), \dots, x+1, k-(x+1), \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{k}{6} \rfloor - 1), \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{6} \rfloor, \lceil \frac{k}{2} \rceil + \lfloor \frac{k}{6} \rfloor, \\ & \dots, \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv x \pmod{k}$ where $k \equiv 0 \pmod{3}$, $x = \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3} + 2, k - (\frac{k}{3} + 2), \frac{k}{3} + 3, k - (\frac{k}{3} + 3), \dots, x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{k}{6} \rfloor - 1), \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{6} \rfloor, \lceil \frac{k}{2} \rceil + \lfloor \frac{k}{6} \rfloor, \\ & \dots, \lceil \frac{k}{2} \rceil + (\lfloor \frac{x}{2} \rfloor - 2), \lceil \frac{k}{2} \rceil - (\lfloor \frac{x}{2} \rfloor - 1) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

for $x = \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \frac{k}{3} + 2, k - (\frac{k}{3} + 2), \frac{k}{3} + 3, k - (\frac{k}{3} + 3), \dots, x, k-x, 0, 1, \\ & \lceil \frac{k}{2} \rceil + (\lfloor \frac{k}{6} \rfloor - 1), \lceil \frac{k}{2} \rceil - \lfloor \frac{k}{6} \rfloor, \lceil \frac{k}{2} \rceil + \lfloor \frac{k}{6} \rfloor, \\ & \dots, \lceil \frac{k}{2} \rceil - (\frac{x}{2} - 2), \lceil \frac{k}{2} \rceil + (\frac{x}{2} - 2) \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 3: If $n \equiv x \pmod{k}$ where $k \equiv 1 \pmod{3}$ and $k > 7$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = \lfloor \frac{k}{3} \rfloor + 2, k - (\lfloor \frac{k}{3} \rfloor + 2), \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv 3 \pmod{k}$ and $k = 7$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor - 1) + 5 & \text{if } i = 0, 1 \\ 3(\lfloor \frac{n}{k} \rfloor - 1) + 4 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv x \pmod{k}$ where $k \equiv 1 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \lfloor \frac{k}{2} \rfloor - 1$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{if } i = \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor - 1, \dots, \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & 0, k-1, 1, k-2, 2, \dots, \frac{x}{2}, k - (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 3 & \text{otherwise.} \end{cases}$$

Subcase 1: If $n \equiv k - x \pmod{k}$ where $k \equiv 0 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\frac{k}{3} - 1), \lfloor \frac{k}{2} \rfloor + \frac{k}{3}, \lfloor \frac{k}{2} \rfloor - \frac{k}{3}, \dots, \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{k}{6} \rfloor + 2), (\lfloor \frac{k}{6} \rfloor + 2), \dots, k - (\lfloor \frac{x}{2} \rfloor + 1), (\lfloor \frac{x}{2} \rfloor + 1), \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 1 \leq x \leq \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\frac{k}{3} - 1), \lfloor \frac{k}{2} \rfloor + \frac{k}{3}, \lfloor \frac{k}{2} \rfloor - \frac{k}{3}, \dots, \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{k}{6} \rfloor + 2), (\lfloor \frac{k}{6} \rfloor + 2), \dots, \frac{x}{2}, k - (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 2: If $n \equiv k - x \pmod{k}$ where $k \equiv 1 \pmod{3}$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor - 1), \lfloor \frac{k}{2} \rfloor + (\lfloor \frac{k}{3} \rfloor) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 3: If $n \equiv k - x \pmod{k}$ where $k \equiv 1 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor - 1), \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor, \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{3} \rfloor, \dots, \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), \dots, k - (\lfloor \frac{x}{2} \rfloor + 1), (\lfloor \frac{x}{2} \rfloor + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor - 1), \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{3} \rfloor, \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{3} \rfloor, \dots, \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), \dots, \frac{x}{2}, k - (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 4: If $n \equiv k - x \pmod{k}$ where $k \equiv 2 \pmod{3}$ and $k > 5$, $x = \lfloor \frac{k}{3} \rfloor + 1$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 5: If $n \equiv k - x \pmod{k}$ where $k = 5$, $x = 2$,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = 2, 3 \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Subcase 6: If $n \equiv k - x \pmod{k}$ where $k \equiv 2 \pmod{3}$, $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \lfloor \frac{k}{2} \rfloor$ and x is odd,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{3} \rfloor, \lfloor \frac{k}{2} \rfloor + (\lfloor \frac{k}{3} \rfloor + 1), \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor + 1), \\ & \dots, \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), \lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2, \dots, k - (\lfloor \frac{x}{2} \rfloor + 1), (\lfloor \frac{x}{2} \rfloor + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

for $\lfloor \frac{k}{3} \rfloor + 2 \leq x \leq \lfloor \frac{k}{2} \rfloor$ and x is even,

$$e_f(i) = \begin{cases} 3(\lfloor \frac{n}{k} \rfloor) + 1 & \text{if } i = \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k}{3} \rfloor, \lfloor \frac{k}{2} \rfloor + (\lfloor \frac{k}{3} \rfloor + 1), \lfloor \frac{k}{2} \rfloor - (\lfloor \frac{k}{3} \rfloor + 1), \\ & \dots, \lfloor \frac{k}{2} \rfloor - (x-2), \lfloor \frac{k}{2} \rfloor + (x-1), \\ & k - (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), (\lfloor \frac{1}{2} \lfloor \frac{k}{3} \rfloor \rfloor + 2), \dots, \frac{x}{2}, k - (\frac{x}{2} + 1) \\ 3(\lfloor \frac{n}{k} \rfloor) + 2 & \text{otherwise.} \end{cases}$$

Therefore, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq k-1$. Hence, the graph $S'(K_{1,n})$ is k -product cordial for $n \geq \lfloor \frac{k}{2} \rfloor$ where k is odd and $k \geq 4$. $\square$

An example of 15-product cordial labeling of $S'(K_{1,11})$ is shown in Figure 3.

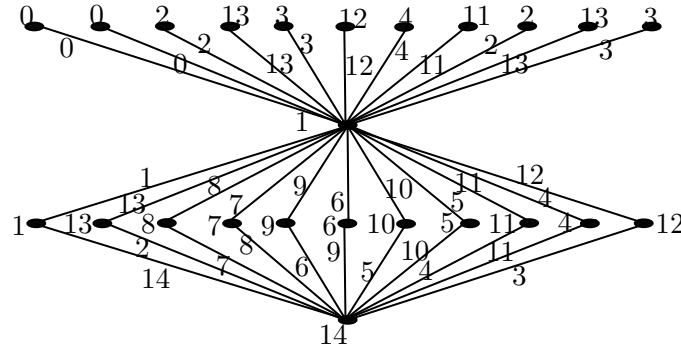


Figure 3: 15-product cordial labeling of $S'(K_{1,11})$

Theorem 2.3. The graph $S'(K_{1,n})$ is k -product cordial if and only if $n \geq 1$ and $k \geq 4$ except $\lfloor \frac{k}{3} \rfloor \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 1, 0 \pmod{3}$ for $k = 4, 6$ and $k \geq 8$ and $\lceil \frac{k}{3} \rceil \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 2 \pmod{3}$ for $k \geq 8$.

Proof. From Theorems 2.1 and 2.2, the graph $S'(K_{1,n})$ is k -product cordial if $n \geq \lfloor \frac{k}{2} \rfloor$ and $k \geq 4$.

For $k \geq 5$, $1 \leq n \leq \lfloor \frac{k}{3} \rfloor - 1$ if $k \equiv 1, 0 \pmod{3}$ and $1 \leq n \leq \lceil \frac{k}{3} \rceil - 1$ if $k \equiv 2 \pmod{3}$.

For k is even and $1 \leq i \leq n$, we assign $f(u) = 1$, $f(v) = k - 1$, $f(u_1) = 0$, $f(u_2) = \frac{k}{2}$ and $f(v_i) = \frac{k}{2} + i$.

For $3 \leq i \leq n$, we assign $f(u_i) = \begin{cases} \frac{i-1}{2} + 1 & \text{if } i \text{ is odd} \\ k - \frac{i}{2} & \text{if } i \text{ is even.} \end{cases}$

For k is odd and $1 \leq i \leq n$, we assign

$f(u) = 1$, $f(v) = k - 1$, $f(u_1) = 0$, and $f(v_i) = \lfloor \frac{k}{2} \rfloor + i$.

For $2 \leq i \leq n$, we assign $f(u_i) = \begin{cases} \frac{i}{2} + 1 & \text{if } i \text{ is even} \\ k - 1 - \frac{i-1}{2} & \text{if } i \text{ is odd.} \end{cases}$

From the above labeling pattern, we have

for $n = 1$ and k is even,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 1, k - 1, 0, \frac{k}{2} + 1 \\ 0 & \text{otherwise} \end{cases}$$

for $n = 2$ and k is even,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 1, k - 1, 0, \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} + 2 \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geq 3$, n and k are even,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 1, k - 1, 2, k - 2, \dots, \frac{n}{2}, k - \frac{n}{2}, \\ & \frac{k}{2} + 1, \frac{k}{2} + 2, \dots, \frac{k}{2} + n \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geq 3$, n is odd and k is even,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 1, k - 1, 2, k - 2, \dots, k - \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \\ & \frac{k}{2} + 1, \frac{k}{2} + 2, \dots, \frac{k}{2} + n \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 1$ and k is odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2} + 1, \frac{k}{2} - 1 \\ 0 & \text{otherwise} \end{cases}$$

for $n = 2$ and k is odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, \frac{k}{2} + 1, \frac{k}{2} - 1, \frac{k}{2} + 2, \frac{k}{2} - 2 \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geq 3$, n and k are odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 2, k - 2, \dots, \frac{n}{2}, k - \frac{n}{2}, \\ & \frac{k}{2} + 1, \frac{k}{2} - 1, \frac{k}{2} + 2, \frac{k}{2} - 2, \dots, \frac{k}{2} + n, \frac{k}{2} - n \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geq 3$, n is even and k is odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \frac{k}{2}, 2, k-2, \dots, k - \lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \\ & \frac{k}{2} + 1, \frac{k}{2} - 1, \frac{k}{2} + 2, \frac{k}{2} - 2, \dots, \frac{k}{2} + n, \frac{k}{2} - n \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 1$ and k is odd,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 1, k-1, 0, \lfloor \frac{k}{2} \rfloor + 1 \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 2$, n and k are odd,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, 1, k-1, 2, k-2, \dots, \lfloor \frac{n}{2} \rfloor + 1, k - (\lfloor \frac{n}{2} \rfloor + 1), \\ & \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor + 2, \dots, \lfloor \frac{k}{2} \rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geq 2$, n is even and k is odd,

$$v_f(i) = \begin{cases} 1 & \text{if } i = 0, 1, k-1, 2, k-2, \dots, k - \frac{n}{2}, \frac{n}{2} + 1, \\ & \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor + 2, \dots, \lfloor \frac{k}{2} \rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$

For $n = 1$ and k is odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1 \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 2$, n and k are odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, 2, k-2, 3, k-3, \dots, \lfloor \frac{n}{2} \rfloor + 1, k - (\lfloor \frac{n}{2} \rfloor + 1), \\ & \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor - 1, \lfloor \frac{k}{2} \rfloor + 2, \dots, \lfloor \frac{k}{2} \rfloor - (n-1), \lfloor \frac{k}{2} \rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$

for $n \geq 2$, n is even and k is odd,

$$e_f(i) = \begin{cases} 1 & \text{if } i = 0, 2, k-2, 3, k-3, \dots, k - \frac{n}{2}, \frac{n}{2} + 1, \\ & \lfloor \frac{k}{2} \rfloor, \lfloor \frac{k}{2} \rfloor + 1, \lfloor \frac{k}{2} \rfloor - 1, \lfloor \frac{k}{2} \rfloor + 2, \dots, \lfloor \frac{k}{2} \rfloor - (n-1), \lfloor \frac{k}{2} \rfloor + n \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $0 \leq i, j \leq k-1$. Hence, the graph $S'(K_{1,n})$ is k -product cordial if $n \geq \lfloor \frac{k}{2} \rfloor$, $1 \leq n \leq \lfloor \frac{k}{3} \rfloor - 1$ for $k \equiv 1, 0 \pmod{3}$ and $1 \leq n \leq \lceil \frac{k}{3} \rceil - 1$ for $k \equiv 2 \pmod{3}$.

Now for $k = 7$ and $n = 2$, we assign $f(u) = 4$, $f(v) = 1$, $f(u_1) = 0$, $f(u_2) = 5$, $f(v_1) = 2$ and $f(v_2) = 3$. From

this labeling, we have $v_f(i) = \begin{cases} 0 & \text{if } i = 6 \\ 1 & \text{otherwise} \end{cases}$

and $e_f(i) = \begin{cases} 0 & \text{if } i = 4 \\ 1 & \text{otherwise} \end{cases}$ for $0 \leq i \leq 6$. Hence, the graph $S'(K_{1,2})$ is 7-product cordial.

Conversely, we assume that the graph $S'(K_{1,n})$ is k -product cordial if $\lfloor \frac{k}{3} \rfloor \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ where $k \equiv 1, 0 \pmod{3}$ for $k \geq 4$ except 7 and $\lceil \frac{k}{3} \rceil \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ for $k \equiv 2 \pmod{3}$ where $k \geq 8$. From this hypothesis, we have $v_f(i) = 1$ or 0, $e_f(i) = 1$ or 0 if $n = \lfloor \frac{k}{3} \rfloor$ or $\lceil \frac{k}{3} \rceil$ and $e_f(i) = 1$ or 2 if $\lfloor \frac{k}{3} \rfloor + 1 \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ or $\lceil \frac{k}{3} \rceil + 1 \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$. Without loss of generality, we assign 1 and $k-1$ to the vertices u and v respectively. Also, we assign $\{0, 2, 3, \dots, k-2\}$ to the remaining vertices u_i and v_i . Now, we have $e_f(1) = e_f(k-1) = 0$, which is a contradiction. Therefore, the graph $S'(K_{1,n})$ is not k -product cordial if $\lfloor \frac{k}{3} \rfloor \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ where $k \equiv 1, 0 \pmod{3}$ for $k \geq 4$ except 7 and $\lceil \frac{k}{3} \rceil \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ for $k \equiv 2 \pmod{3}$ where $k \geq 8$. Hence, the graph $S'(K_{1,n})$ is k -product cordial if and only if $n \geq 1$ and $k \geq 4$ except $\lfloor \frac{k}{3} \rfloor \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 1, 0 \pmod{3}$ for $k = 4, 6$ and $k \geq 8$ and $\lceil \frac{k}{3} \rceil \leq n \leq \lfloor \frac{k}{2} \rfloor - 1$ if $k \equiv 2 \pmod{3}$ for $k \geq 8$. $\square$

An example of 18-product cordial labeling of $S'(K_{1,5})$ shown in Figure 4.

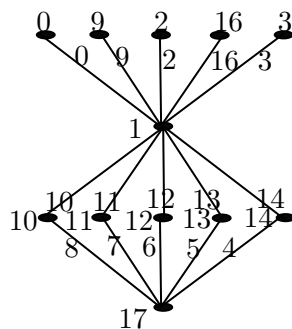


Figure 4: 18-product cordial labeling of $S'(K_{1,5})$

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